Multiple Integral

Double Integrals over Rectangles

Remark：

1. Let $P = \{x_0, x_1, \ldots, x_n\}$ and $a = x_0 < x_1 < \cdots < x_n = b$

   Then P is called a partition of [a , b]

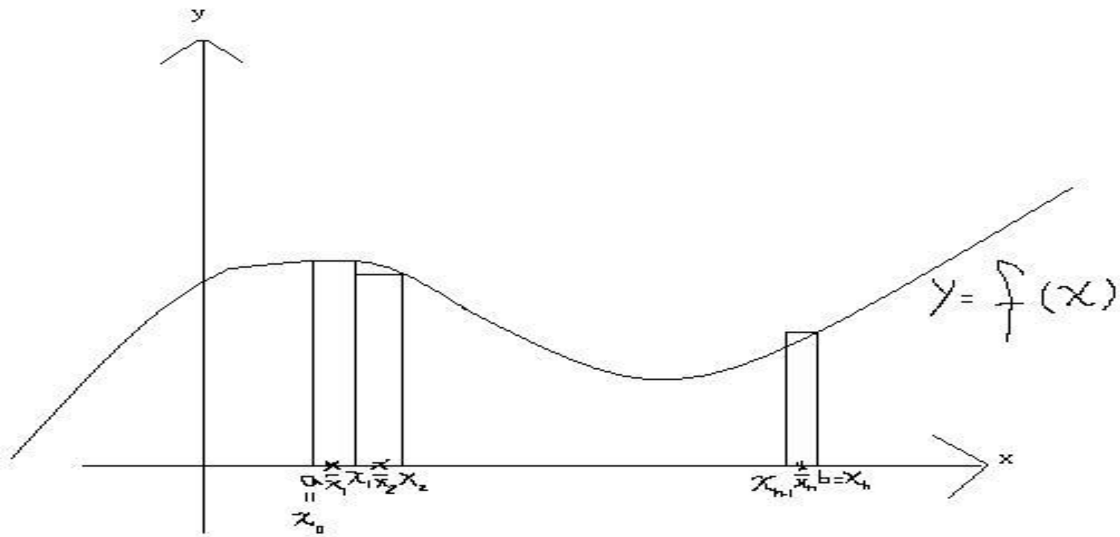
2. Define $\Delta x_i = x_i - x_{i-1},\ i = 1, 2, \ldots, n$

3. $|P| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ The norm of P

4. Choose $\bar{x}_i \in [x_{i-1}, x_i],\ i = 1, 2, \ldots n$, $\bar{x}_i$ is called a sample point. (取樣點)

5. The definite integral of f on [a , b]

$$\int_a^b f(x)dx = \lim_{|P|\to 0}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \text{The area of S}$$



$$S = \{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}$$

5. The definite integral of f on [a , b]

$$\int_a^b f(x)dx = \lim_{|P|\to 0}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i = \text{The area of S}$$

$$S = \{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}$$

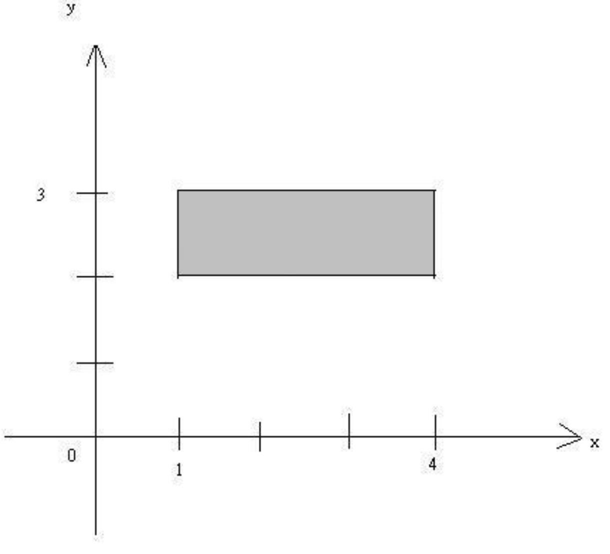
6. A closed rectangle R

$$R = [a, b] \times [c, d] = \{(x, y) \in R^2 \mid a \le x \le b, c \le y \le d\}$$

Example : Rectangle

1. $[0,1] \times [2,4] = \{(x, y) \in R^2 \mid 0 \le x \le 1, 2 \le y \le 4\}$

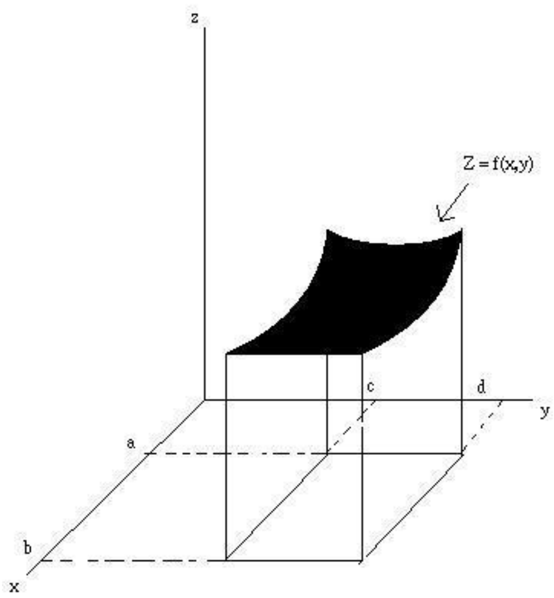
2. $[1,2] \times [2,3]$



Double Integrals and Volumes

$\text{Let } R = [a, b] \times [c, d], S = \{(x, y, z) \in R^3 \mid (x, y) \in R, 0 \le z \le f(x, y)\}$

To find the volume of S - V(S)



Similarly to define $\int_a^b f(x)dx$

Divide the rectangle R into subinterva ls

$[a, b]$ is divided into m subinterval $[x_{i-1}, x_i]$

$i = 1, 2, \cdots m$, $\Delta x_i = x_i - x_{i-1}$, and $[c, d]$ is divided

into n subinterval $[y_{j-1}, y_j]$, $\Delta y_j = y_j - y_{j-1}$, $j = 1, 2, \cdots n$

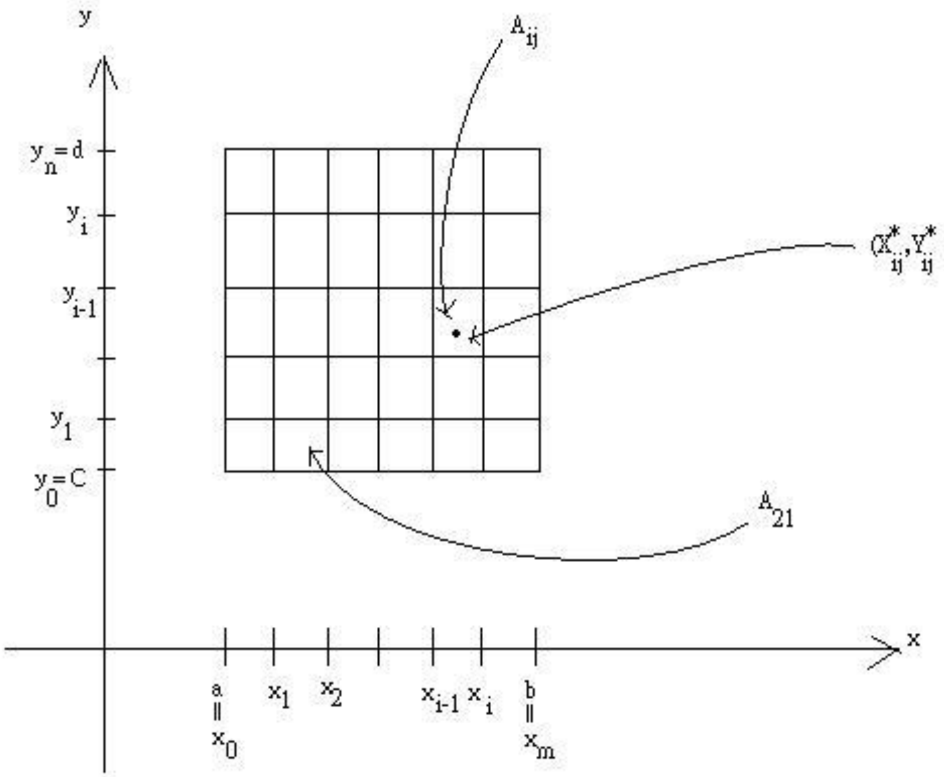
Define $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ $i = 1, \cdots m$; $j = 1, \cdots n$

The area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$

choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij}

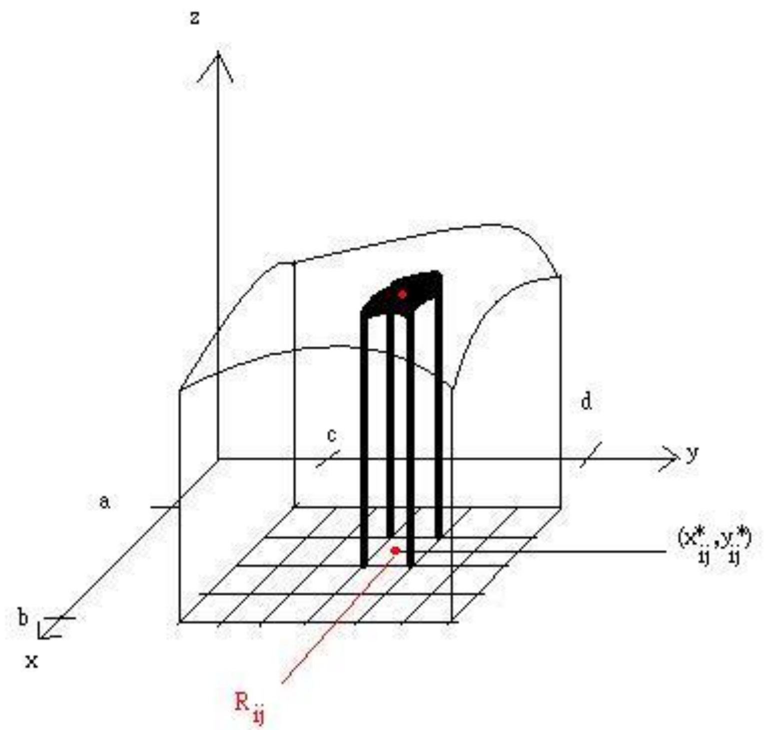
The volume of S can approximate $\sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$

i.e $V(S) \approx \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$



Let $|P|$ denote the length of the longest diagonal

R_{ij}, $i = 1,2 \cdots m$; $j = 1,2 \cdots n$



Definition :

The double integral of f over the rectangle R is $\iint_R f(x, y)dA$

$$\iint_R f(x, y)dA = \lim_{|P|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A_{ij}$$

if this limit exists

properties :

1. $$\iint_R cf(x, y)dA = c\iint_R f(x, y)dA$$

2. $$\iint_R (f(x, y) + g(x, y))dA = \iint_R f(x, y)dA + \iint_R g(x, y)dA$$

3. If $f(x, y) \geq g(x, y)\ \forall (x, y) \in R$, then

$$\iint_R f(x, y)dA \geq \iint_R g(x, y)dA$$

Definition :

1. f is integrable on R, if $\lim\limits_{|P|\to 0} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta A_{ij}$ exist

2. $\iint\limits_{R} f(x, y)\,dA$ is called the double integral of f over R

Theorem 1

Let f be bounded on the closed rectabgle R

(i) If f is continuous on R, then f is integrable on R

(ii) If f is continuous on R except on a finite number of smooth curves, then f is integrable on R

Definition :

1. If $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, then f is continuous at (a, b)

2. If f is continuous at all $(a, b) \in R$, then f is continuous on R

Example :

1. $f(x, y) = \sin xy, \ (x, y) \in R = [0, \pi] \times [0, 2\pi]$

   f is continuous on R

2. $f(x, y) = x^2 y + x, \ R = [0, \infty) \times [0, \infty)$

   f is continuous on R

3. $f(x, y) = \begin{cases} \dfrac{y}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$

   f is not continuous on (0,1)

   f is not continuous on $(o, w), \forall w$

Iterated Integrals

$$\text{Fixed } y, \text{ Let } A(y) = \int_1^3 x^2 y\,dx = y \cdot \frac{x^3}{3}\Bigg|_1^3 = \frac{26}{3}y$$

$$\text{Consider } \int_0^2 A(y)\,dy = \int_0^2 \left(\int_1^3 x^2 y\,dx\right)dy$$

Remark :

For function $f(x, y)$, $R = [a, b] \times [c, d]$

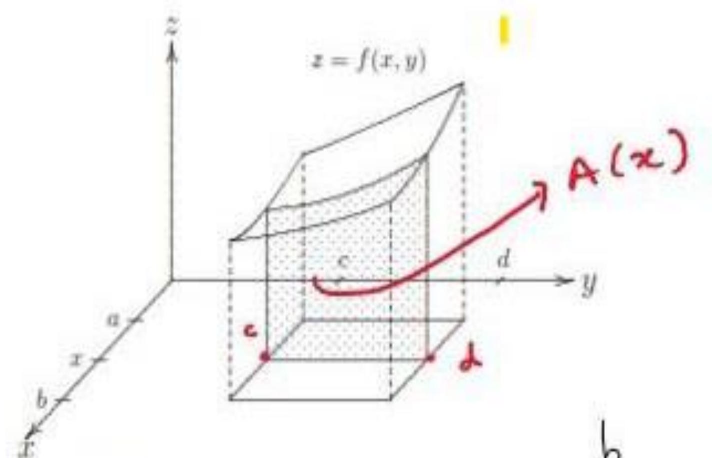
1. Let $A(x) = \int_c^d f(x, y)\,dy$, $B(y) = \int_a^b f(x, y)\,dx$

$\int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x, y)\,dy\right)dx$ is called an iterated integral

Iterated Integrals

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)\, dy$ to mean that x is held fixed (and treated as a constant) and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)\, dy$$



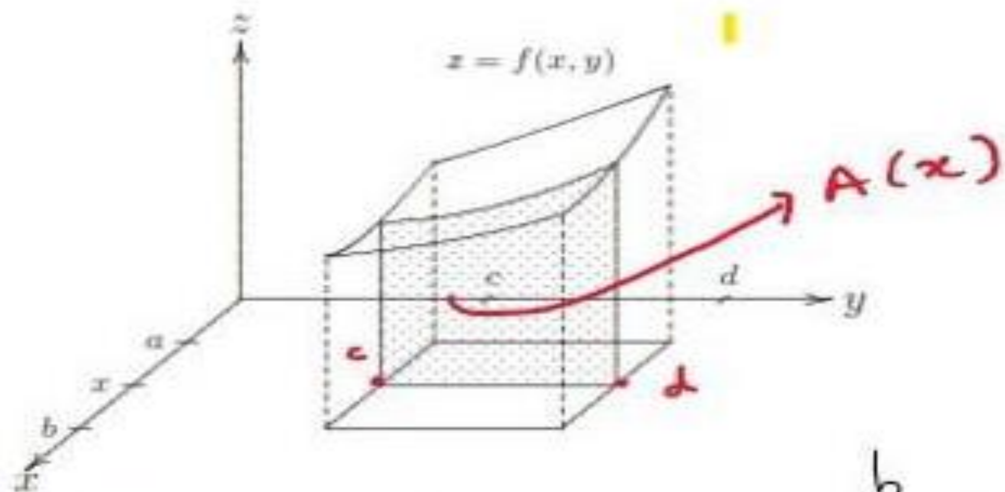
3 $\quad \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b A(x)\,dx$

Iterated Integrals

If we now integrate the function A with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx \qquad (3)$$

The integral on the right side of Equation 7 is called an **iterated integral**. Usually the brackets are omitted. Thus



$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b A(x)\,dx \qquad (3)$$

Iterated Integrals

Similarly, the iterated integral

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$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

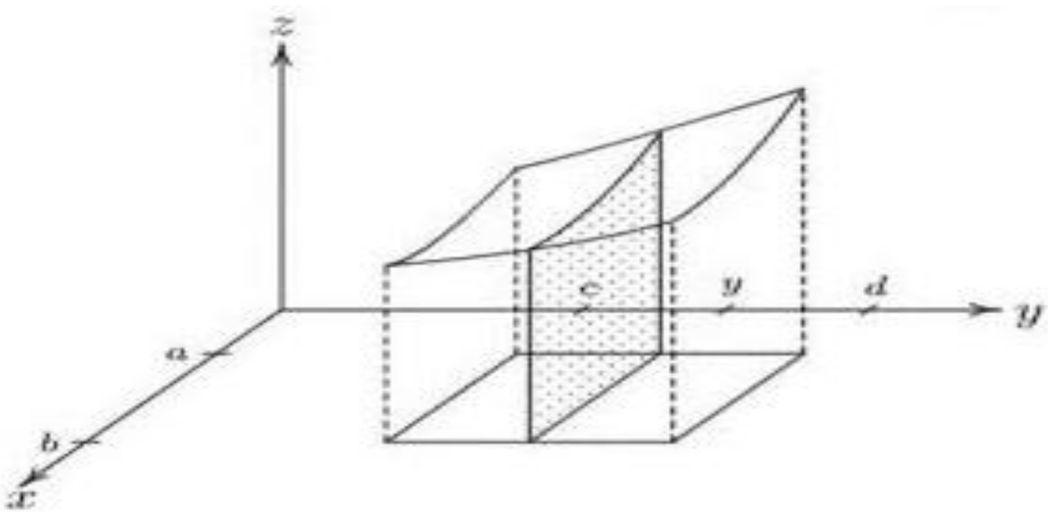


Figure 68 $\int_c^d \int_a^b f(x, y) dx dy$

2. $\int_a^b \int_c^d f(x, y)dydx = \int_a^b \left(\int_c^d f(x, y)dy\right)dx$

3. $\int_c^d \int_a^b f(x, y)dxdy = \int_c^d \left(\int_a^b f(x, y)dx\right)dy$

Example : evaluate

(i) $\int_0^3 \int_1^4 x^2 ydxdy$ (ii) $\int_0^3 \int_1^4 x^2 ydydx$

(iii) $\int_0^4 \int_0^8 \frac{1}{4}(64 - 8x + y^2)dydx$

(iv) $\int_0^8 \int_0^4 \frac{1}{4}(64 - 8x + y^2)dxdy$

(v) $\int_0^3 \int_1^2 (3x + 2y)dxdy$

Theorem 2 (Fubini' s Theorem)

If f is continuous on $R=[a,b]\times[c,d]$, the

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

Ex :

1. Find $\iint_R (x - 3y^2)dA$, where $R=[0,2]\times[1,3]$

2. Find $\iint_R y\sin xy\,dA$, where $R=[0,2]\times[0,\pi]$

Ans : $\iint_R y\sin xy\,dA = \int_0^2 \int_0^{\pi} y\sin xy\,dydx = ?$

But $\int_0^{\pi} \int_0^2 y\sin xy\,dxdy = \int_0^{\pi} (-\cos xy)\Big|_0^2 dy = \int_0^{\pi} (1-\cos 2y)dy$

$= \pi - (\frac{1}{2}\sin 2y\Big|_0^{\pi}) = \pi - 0 = \pi$

3. Find $\iint_R x\cos xy\,dA$, where $R=[0,\pi]\times[0,\pi]$

4. Find $\iint_R \frac{1+y}{1+x}dA$, where $R=\{(x, y) \mid -1\le x \le 2, 0 \le y \le 1\}$

Exercises

1. Let $R = \{(x, y) \mid 1 \le x \le 3,\ 0 \le y \le 2\}$, Evaluate $\iint_R f(x, y)\,dA$, where f is the given function

   1. $f(x, y) = \begin{cases} 2, & 1 \le x \le 3,\ 0 \le y \le 2 \\ 3, & 3 < x \le 4,\ 0 \le y \le 2 \end{cases}$

   2. $f(x, y) = \begin{cases} 2, & 1 \le x \le 4,\ 0 \le y \le 1 \\ 1, & 1 \le x < 3,\ 1 \le y \le 2 \\ 4, & 3 \le x \le 4,\ 1 \le y \le 2 \end{cases}$

2. If $R = [0,1] \times [0,1]$, show that

$$D \le \iint_R \sin(x + y)\,dA \le 1$$

3. Let $R_1 = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 1\}, R = [0,2] \times [0,2]$

$R_2 = \{(x, y) \mid 0 \le x \le 2, 1 \le y \le 2\}$

Suppose that $\iint_R f(x, y)dA = 4, \iint_R g(x, y)dA = 6, \iint_{R_2} g(x, y)dA = 2$

Evaluate

1. $\iint_R (4f(x, y) - g(x, y))dA$

2. $\iint_{R_1} 3g(x, y)dA$

3. $\iint_R (2f(x, y) + 4)dA$

4. $\iint_R (3f(x, y) + 4g(x, y))dA$

4. Evaluate each of the interated integrals

(i) $\int_0^2 \int_1^3 f(x, y)dydx$, where $f(x, y) = x^2 y$

(ii) $\int_0^1 \int_0^1 xe^{xy} dxdy$

(iii) $\int_0^{\pi} \int_0^1 x\sin y dxdy$

(iv) $\int_0^2 \int_0^2 \frac{y}{1+x^2} dydx$

(v) $\int_{-2}^2 \int_{-1}^1 |x^2 y^3| dydx$

(vi) $\int_{-2}^2 \int_{-1}^1 [x^2] y^3 dydx$

(vii) $\int_{-2}^2 \int_{-1}^1 [x] |y| dydx$

5. Calculate the double integral

(i) $\iint_R (2y^2 - 3xy^3)\,dA,\ R = \{(x, y) | 1 \le x \le 2,\ 0 \le y \le 3\}$

(ii) $\iint_R x\cos(x + y)\,dA,\ R = [0, \frac{\pi}{6}] \times [0, \frac{\pi}{3}]$

(iii) $\iint_R \frac{1+x}{2+y}\,dA,\ R = \{(x, y) | -1 \le x \le 2,\ 0 \le y \le 1\}$

(iv) $\iint_R xye^{x^2y^2}\,dA,\ R = [0,1] \times [0,1]$

(v) $\iint_R xy\sqrt{1+x^2}\,dA,\ R = \{(x, y) | 0 \le x \le \sqrt{3}, 1 \le y \le 2\}$

6. Find the volume of the solid lying under the plane

$Z = 2x + 5y + 1$ and above the rectangle

$\{(x, y) \mid (-1 \le x \le 0, 1 \le y \le 4)$

7. Evaluate the interated integral

(i) $\int_0^1 \int_0^x \cos x^2 \, dydx$ (ii) $\int_0^1 \int_{1-x}^{2+x} (3x - y^2) dydx$

(iii) $\int_0^1 \int_{3y}^3 e^{x^2} dxdy$ (iv) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1 + x^3} \, dxdy$

(v) $\int_0^1 \int_0^{y^2} 2ye^x dxdy$ (vi) $\int_0^4 \int_{x/2}^3 e^{y^2} dydx$

8. Evaluate $\iint_D \sqrt{1 - x^2 - y^2}\, dA$, where $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$

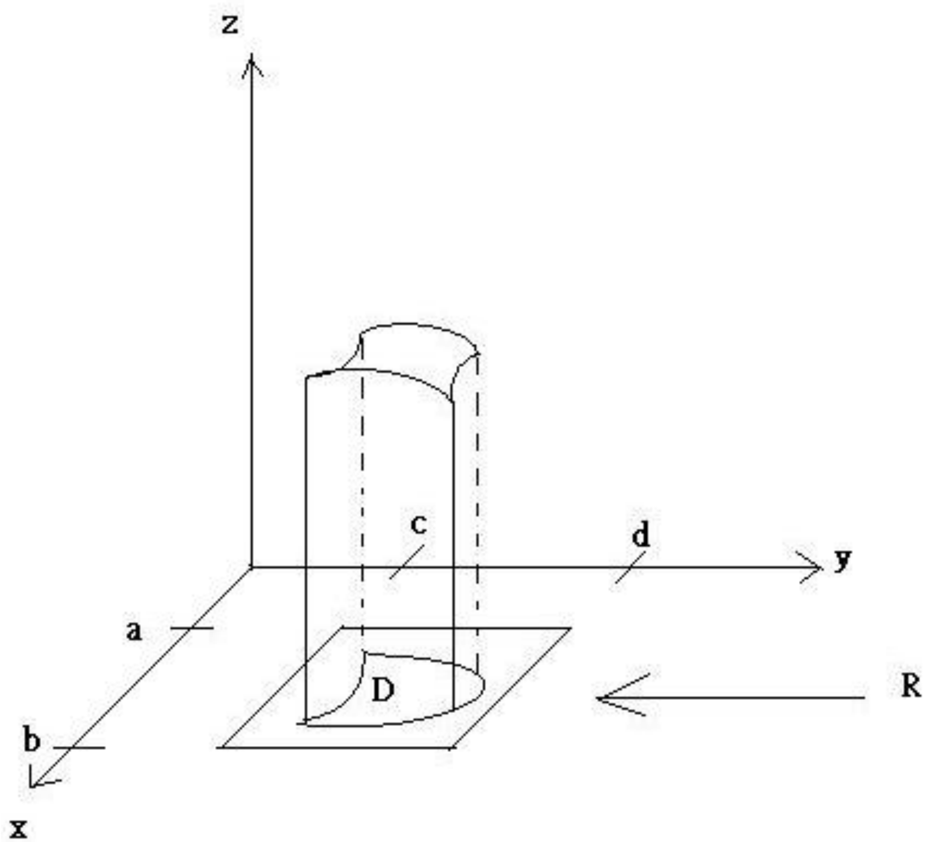
9. Find the volume of the given solid

(i) Bounded by the cylinders $x^2 + y^2 = 2^2$ and $y^2 + z^2 = 2^2$

(ii) Bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$

Double Integral over General Regions

Let D be a bounded region and $D \subset R$, f is a function defined on D. Define a new function



$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 \text{ if } (x, y) \text{ is in R but not in D} \end{cases}$$

Definition :

1. The double integral of f over D is

$$\iint_D f(x, y)\,dA = \iint_R F(x, y)\,dA$$

2. A plane region D is said to be of type I if

$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$,

where g_1, g_2 are two continuous function.

3. A plane region D is said to be of type II if

$D = \{(x, y) \mid h_1(y) \le x \le h_2(y),\ c \le y \le d\}$,

where h_1, h_2 are two continuous function.

Example :

1. $D_1 = \{(x, y) \mid 0 \le x \le \pi, \sin x \le y \le 1\}$, Type I

2. $D_2 = \{(x, y) \mid -1 \le y \le 1, 2y^2 \le x \le 1 + y^2\}$, Type II

Properties :

1. If f is continuous on a type I region D such that

   $D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$,

   then $\iint_D f(x, y)\,dA = \int_a^b \int_{g_{1(x)}}^{g_2(x)} f(x, y)\,dy\,dx$

2. If f is continuous on a type II region D then

   $\iint_D f(x, y)\,dA = \int_c^d \int_{h_{1(y)}}^{h_2(y)} f(x, y)\,dx\,dy$

   where $D = \{(x, y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$

Example :

1. Evaluate $\iint_D (x+3y)dA$

Where $D = \{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2\}$

Ans :

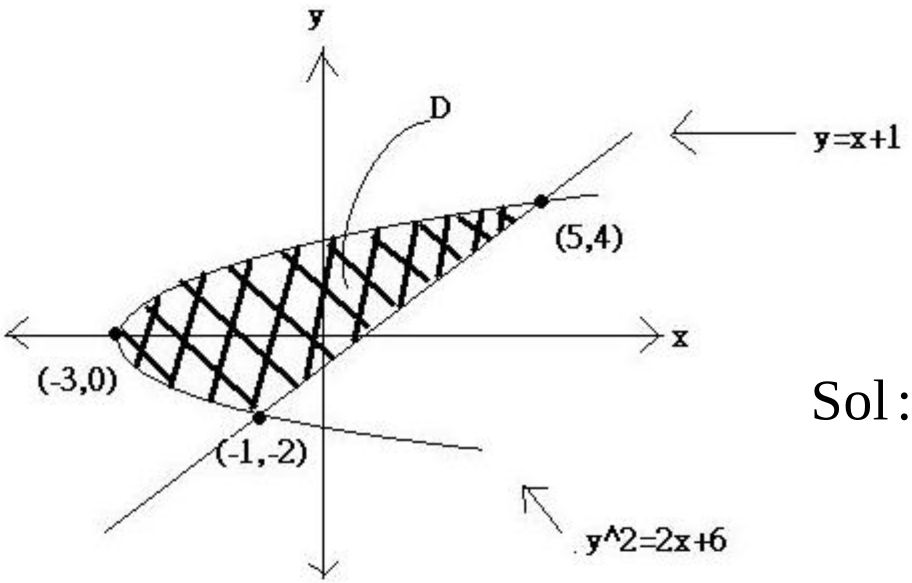
$$\iint_D (x+3y)dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+3y)dydx$$

$$= \int_{-1}^{1} x(1+x^2-2x^2)+\frac{3}{2}((1+x^2)^2-(2x^2)^2)dx$$

$$= \int_{-1}^{1} x+x^3-2x^3+\frac{3}{2}+3x^2+\frac{3}{2}x^4-4x^4dx$$

$$= (\frac{1}{2}x^2-\frac{1}{4}x^4+\frac{3}{2}x+x^3-\frac{1}{2}x^5)\Big|_{-1}^{1} = \frac{3}{2}+1-\frac{1}{2} = 2$$

2. Evaluate $\iint_D xy\,dA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$



Sol:

$$D = \{(x, y) \mid -3 \le x \le 5, ? \le y \le \sqrt{2x+6}\}$$

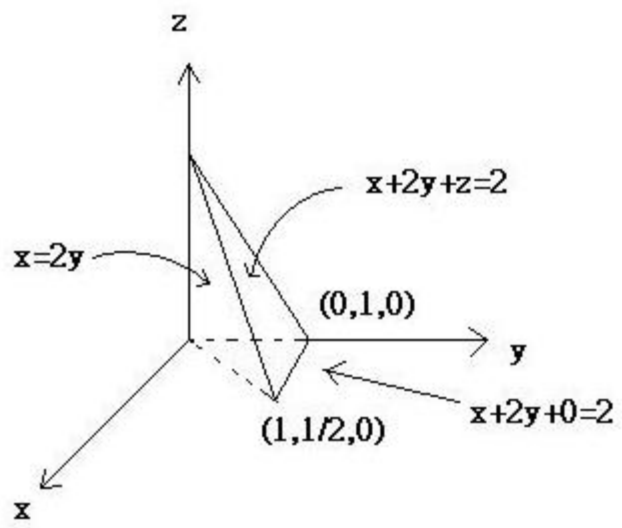
$$= \{(x, y) \mid \frac{y^2 - 6}{2} \le x \le y + 1, -2 \le y \le 4\}$$

$$\iint_D xy\,dA = \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy\,dx\,dy = 36$$

3. Find the volume of the tetrahedron bounded by the planes

$x = 2y,\ x = 0,\ z = 0$ and $x + 2y + z = 2$

Sol:



$$D = \{(x, y) \mid 0 \le x \le 1, \frac{x}{2} \le y \le \frac{2-x}{2}\}$$

$$\text{所求 } V = \iint_D 2 - x - 2y\,dA = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2 - x - 2y)\,dy\,dx$$

$$= \frac{1}{3}$$

4. Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dydx$; $\quad \frac{1}{2}(1-\cos 1)$

5. Evaluate $\int_0^1 \int_x^1 \cos(y^2)\,dydx$

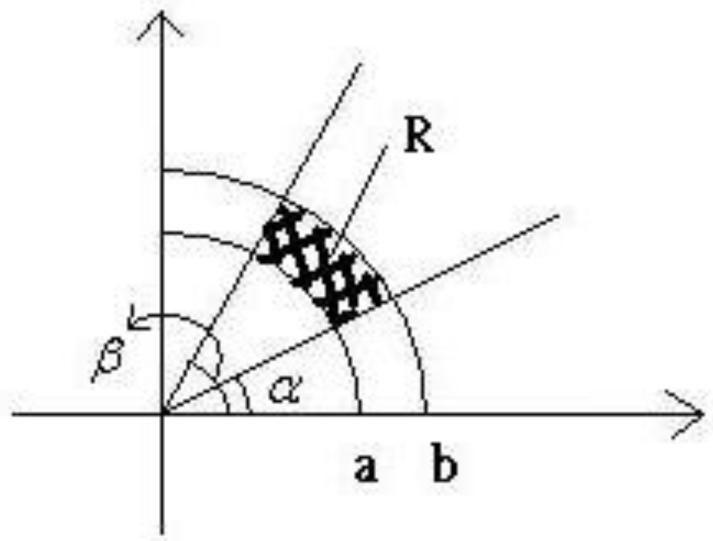
Sol:

$$D = \{(x, y) \mid 0 \le x \le 1,\ x \le y \le 1\}$$

$$= \{(x, y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$$

Double Integrals in Polar Coordinates

Consider $R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta$



Polar rectangle

Example :

1. $R = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

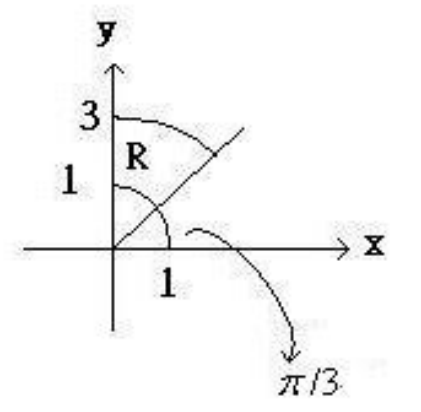
2. $R = \{(r, \theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}$

3. $R = \{(r, \theta) \mid 1 \le r \le 3, \frac{\pi}{3} \le \theta \le \frac{\pi}{2}\}$

$$\text{The area of R is } A(R) = (\pi \cdot 3^2 - \pi \cdot 1^2)\frac{\frac{\pi}{2} - \frac{\pi}{3}}{2\pi}$$

$$= \frac{1}{2}(3^2 - 1^2) \cdot (\frac{\pi}{2} - \frac{\pi}{3})$$

$$= \frac{2}{3}\pi$$



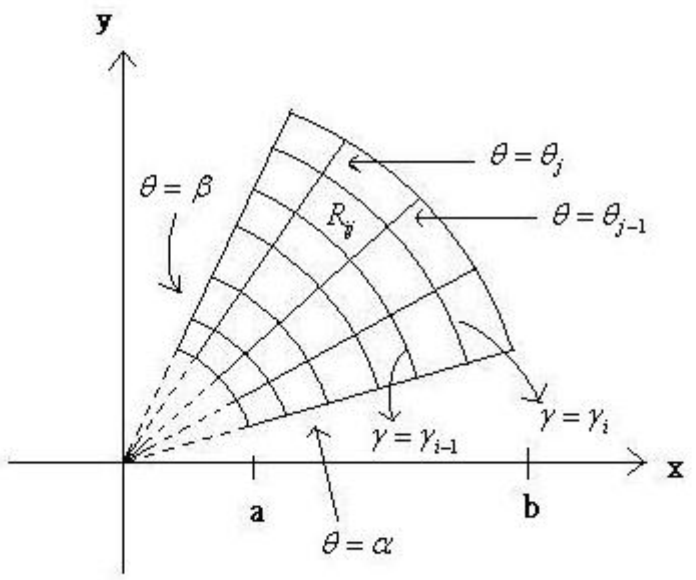
4. $R_{ij}=\{(r,\theta)\,|\,r_{i-1}\le r\le r_i,\ \theta_{j-1}\le\theta\le\theta_j\}$

The area of R_{ij} - ΔA_{ij} is

$$\Delta A_{ij}=\frac{1}{2}r_i^2\Delta\theta_j-\frac{1}{2}r_{i-1}^2\Delta\theta_j=\frac{1}{2}(r_i+r_{i-1})(r_i-r_{i-1})\Delta\theta_j$$

$$=r_i^*\Delta r_i\Delta\theta_j$$

Where $\Delta r_i=r_i-r_{i-1},\ \Delta\theta_j=\theta_j-\theta_{j-1},\ i=1,\cdots m;\ j=1,\cdots n$



The Riemanu sum of f on R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n}f(r_i^*\cos\theta_j^*,r_i^*\sin\theta_j^*)\Delta A_{ij}$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}f(r_i^*\cos\theta_j^*,r_i^*\sin\theta_j^*)r_i^*\Delta r_i\Delta\theta_j$$

$$\to\int_{\alpha}^{\beta}\int_{a}^{b}f(r\cos\theta,r\sin\theta)r\,dr\,d\theta$$

Properties

1. Let $R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$ be a polar rectangle and $0 \le \beta - \alpha \le 2\pi$ If f is continuous on R, then

$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

2. Let $D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ be a polor region. If f is continuous on D then

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Example :

1. Evaluate $\iint_R (4y^2 + 3x)\,dA$

where $R = \{(x, y) \mid y \ge 0, 1 \le x^2 + y^2 \le 4\}$

Sol :

$$R = \{(x, y) \mid y \ge 0, 1 \le x^2 + y^2 \le 4\}$$
$$= \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$$

$$\iint_R (4y^2 + 3x)\,dA = \int_0^{\pi}\int_1^{2} (4(r\sin\theta)^2 + 3r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} (15\sin^2\theta + 7\cos\theta)\,d\theta$$

$$= \frac{15}{2}\pi$$

2. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$

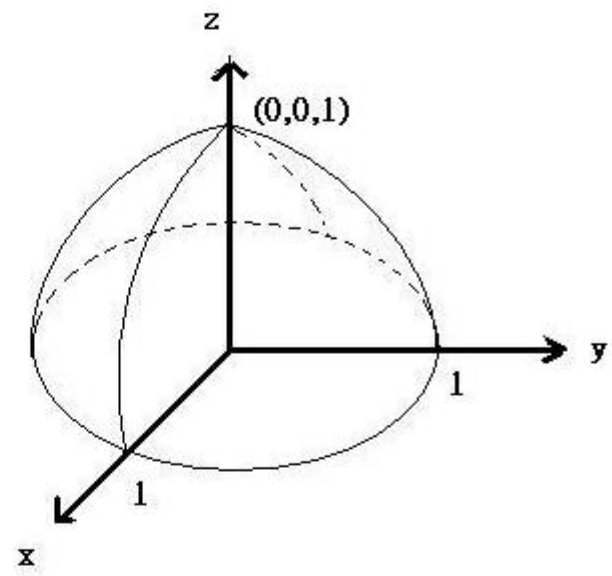
Sol:

$$D = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$$

$$V = \iint_D (1 - x^2 - y^2)\,dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\,r\,dr\,d\theta$$

$$= \frac{\pi}{2}$$



Example :

$$\text{Evaluate } \iint_{R^2} e^{-(x^2+y^2)}\,dA$$

$$\text{where } R^2 = \{(x, y) \mid -\infty < x < \infty, -\infty < y < \infty\}$$

Sol :

$$\text{Consider } D_n = \{(r, \theta) \mid 0 \le r \le n, 0 \le \theta \le 2\pi\}$$

$$\iint_{R^2} e^{-(x^2+y^2)}\,dA = \lim_{n\to\infty} \iint_{D_n} e^{-(x^2+y^2)}\,dA$$

$$= \lim_{n\to\infty} \int_0^{2\pi}\int_0^n e^{-r^2} r\,dr\,d\theta = \lim_{n\to\infty} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}e^{-n^2}\right) d\theta$$

$$= \pi$$

The Cross Product

Definition

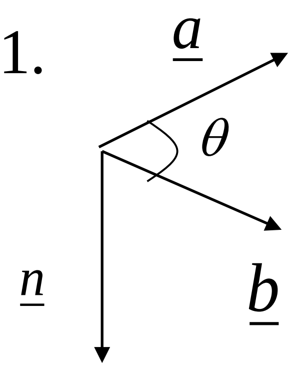
Let a, b be two nonzero three dimensional vectors

1. The inner product of $\underline{a}$ and $\underline{b}$ is $\underline{a} \cdot \underline{b} = |a||b|\cos\theta$

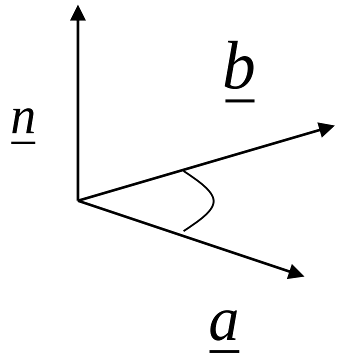
2. The cross product of $\underline{a}$ and $\underline{b}$ is the vector $\underline{a} \times \underline{b} = (|\underline{a}||\underline{b}|\sin\theta)\underline{n}$ where θ is the angle between $\underline{a}$ and $\underline{b}$, $0 \le \theta \le \pi$, and $\underline{n}$ is a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$ and whose direction is given by the right-hand rule: If the fingers of your right hand curl through the angle θ from $\underline{a}$ to $\underline{b}$, then your thumb points in the direction of $\underline{n}$

Example :

1.



2.



Properties

1. $\underline{a}$ and $\underline{b}$ are parallel if and only if $\underline{a} \times \underline{b} = \underline{0}$

2. $i = (1,0,0),\ j = (0,1,0),\ k = (0,0,1)$

(i) $i \times j = k,\ j \times i = -k$

(ii) $j \times k = i,\ k \times j = -i$

(iii) $k \times i = j,\ i \times k = -j$

3. $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$

4. Let c bea$_3$ a scalar

(i) $(c\underline{a})\times\underline{b} = c(\underline{a}\times\underline{b}) = \underline{a}\times(c\underline{b})$

(ii) $\underline{a}\times(\underline{b}+\underline{D}) = \underline{a}\times\underline{b}+\underline{a}\times\underline{D}$

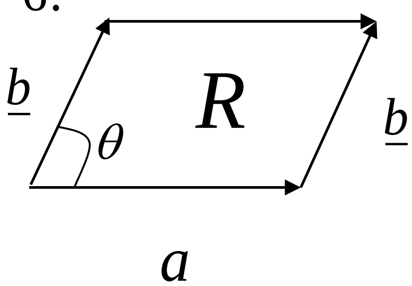
(iii) $(\underline{a}+\underline{b})\times\underline{D} = \underline{a}\times\underline{D}+\underline{b}\times\underline{D}$

5. If $\underline{a} = (a_1, a_2, a_3)$, $\underline{b} = (b_1, b_2, b_3)$, then

$$a\times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

6.



The area of R is

$$A(R) = |\underline{a}\times\underline{b}|$$

Example

1. $\underline{a} = (1,2,0), \underline{b} = (2,-1,3)$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 6i - 3j - 5k$$

$$\underline{b} \times \underline{a} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & 2 & 0 \end{vmatrix} = -6i + 3j + 5k$$

2. $\underline{a} = (1,3,4), \underline{b} = (2,7,-5)$

$\underline{a} \times \underline{b} = -43i + 13j + k$

3. $\underline{a} = i + j + k, \underline{b} = i + j - k$, Find $\underline{a} \times \underline{b}$

4. Find two unit vecto rs orthogonal to both $i + j$ and $i - j + k$

Surface Area

Definition :

$$1.\ \text{parametric curve}: \begin{cases} x = x(t) \\ y = y(t) \end{cases}, \alpha \le t \le \beta$$

$$2.\ \text{The set of all points } (x, y, z) \in R^3$$

$$\text{such that } \begin{cases} x = x(u, v) \\ y = y(u, v), (u, v) \in D \subset R^2 \\ z = z(u, v) \end{cases}$$

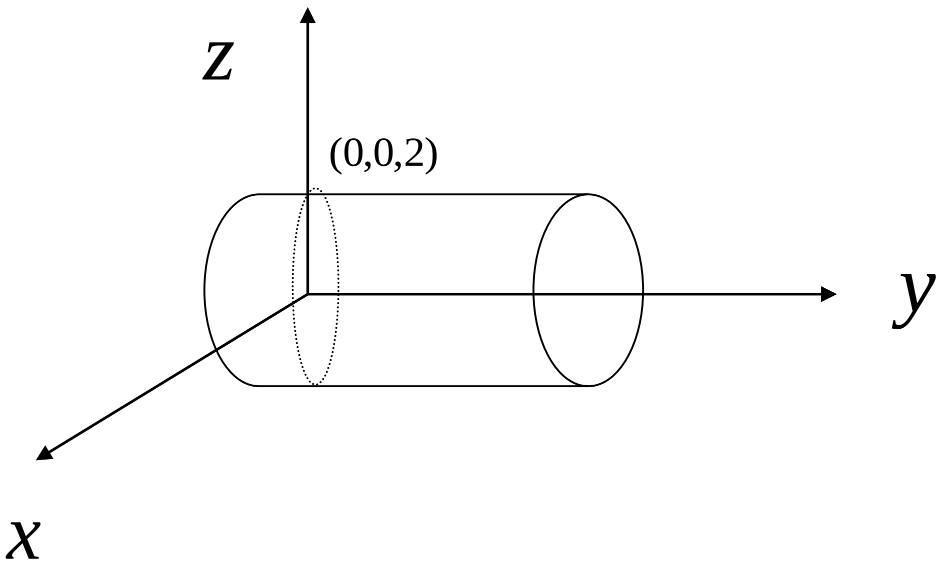
is called a parametric surface S

Example :

1. $S = \{(x, y, z) \mid x = 2\cos u,\ y = v,\ z = 2\sin u\}$

S is a parametric surface

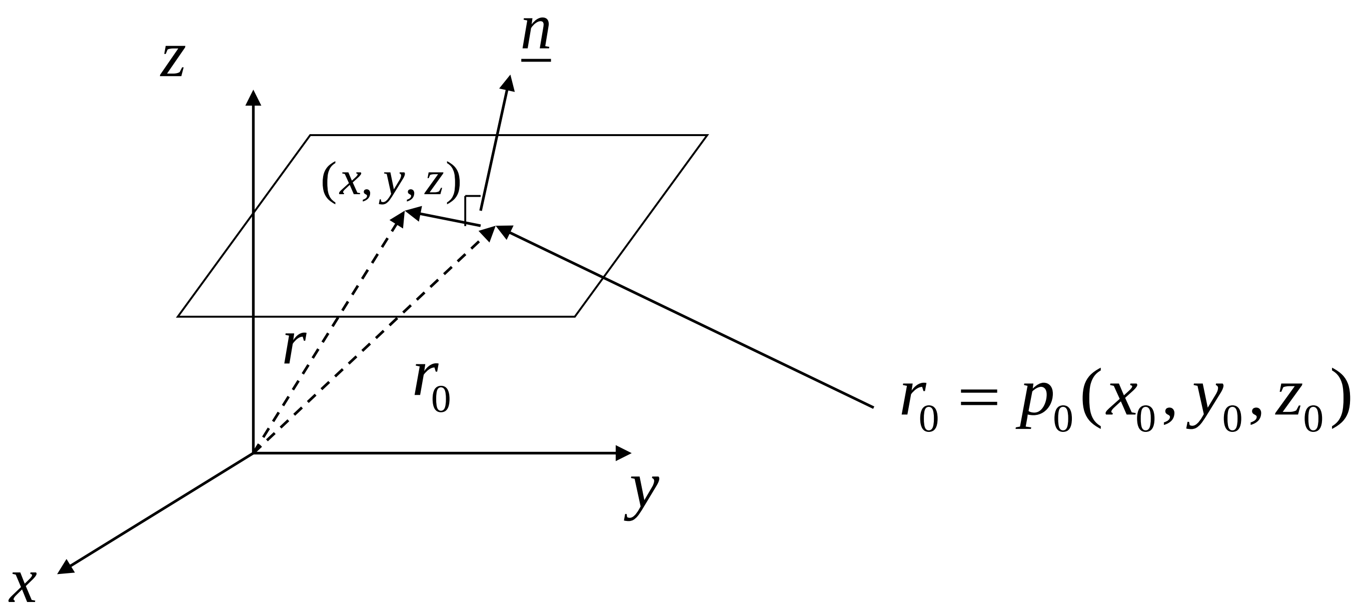
For any $(x, y, z) \in S$, we have $x^2 + z^2 = 4,\ y = v,\ v \in R$



Definition :

3. A plane in space is determined by a point $r_0 = P_0(x_0, y_0, z_0)$ in the plane and a vector $\underline{n}$ is orthogonal to the plane.
This orthogonal vector $\underline{n}$ is called a normal vector

4. The plane is denoted by $\underline{n} \cdot (r - r_0) = \underline{0}$ where $r = (x, y, z)$
n is a normal vector of the plane
A vector equation of the plane $\underline{n} \cdot (r - r_0) = \underline{0}$



Example :

$1.\ S=\{(x,y,z)\mid(1,2,4)\cdot(x-3,y+1,z-4)=0\}$

$2.\ S=\{(x,y,z)\mid 2x-4y+z=0\},\ \underline{n}=(2,-4,1)$

$3.\ S=\{(x,y,z)\mid x+2y+3z=6\},\ \underline{n}=(1,2,3)$

Example :

Find an equation of the plane that passes through the points $P(1,3,2), Q(3,-1,6), R(5,2,0)$

Sol : $12(x-1)+20(y-3)+14(z-2)=0$

Definition :

5. Two plane are parallel if their normal vectors are parallel

6. The angle between the planes S_1 and S_2 is θ if their normal vectors have the angle θ

Example :

Find the angle between the plane $x + y + z = 1$ and $x - 2y + 3z = 2$

Sol :

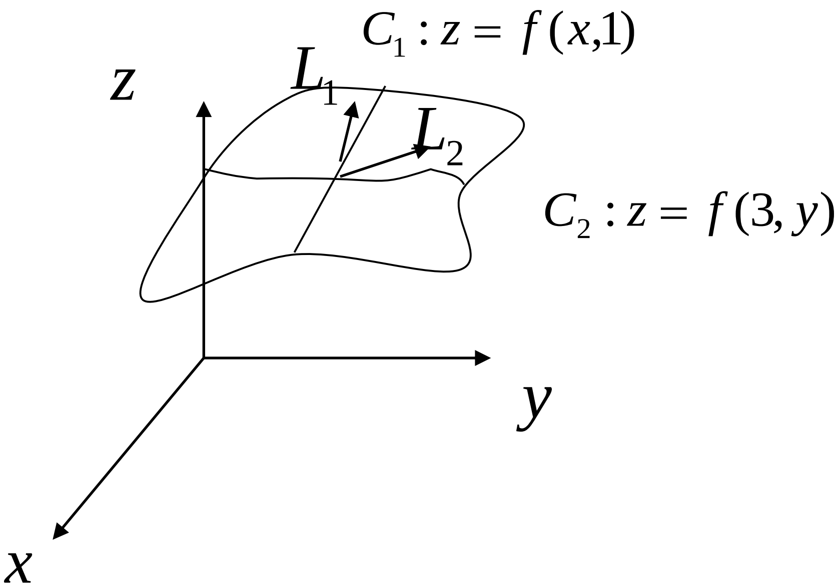
The normal vectors of these planes are $\underline{n}_1 = (1,1,1)$, $\underline{n}_2 = (1,-2,3)$

Let θ be the angle between the planes

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{2}{\sqrt{42}}, \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

Example :

$$z = f(x, y) = x^2 y, \quad \frac{\partial f(x,1)}{\partial x} = 2x, \quad \frac{\partial f(3, y)}{\partial y} = 9$$



$$L_i, i = 1,2, \text{ is the tangent line of } C_i$$

Definition :

Let S be a parametric surface and be defined by

$S = \{(x, y, z) \mid z = z(u, v), y = y(u, v), x = x(u, v)\}$

$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$ Fixed (u_0, v_0)

(i) $r(u_0, v)$ is a vector function of the single parameter v and defines a grid curve c_1 lying on S

(ii) $r(u, v_0)$ is a vector function of the single parameter u and defines a grid curve c_2 lying on S

(iii) The tangent v ector to c_1 at $P_0 - (x_0, y_0, z_0)$ where $x_0 = x(u_0, v_0)$ $y_0 = y(u_0, v_0), z_0 = z(u_0, v_0)$ is obtained

$$r_v = \frac{\partial x(u_0, v_0)}{\partial v} i + \frac{\partial y(u_0, v_0)}{\partial v} j + \frac{\partial z(u_0, v_0)}{\partial v} k$$

(iv) Simlarly t o c_2 The tangent v ector is

$$r_u = \frac{\partial x(u_0, v_0)}{\partial u} i + \frac{\partial y(u_0, v_0)}{\partial u} j + \frac{\partial z(u_0, v_0)}{\partial u} k$$

(v) The surface S is called smooth if $r_u \times r_v \neq \underline{0}$

Exercises 2

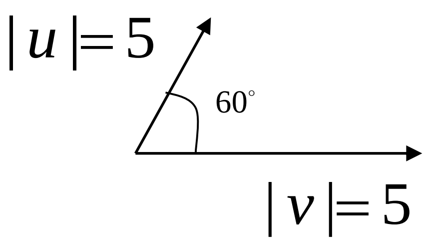
1. Find the cross product $\underline{a} \times \underline{b}$, $\underline{b} \times \underline{a}$

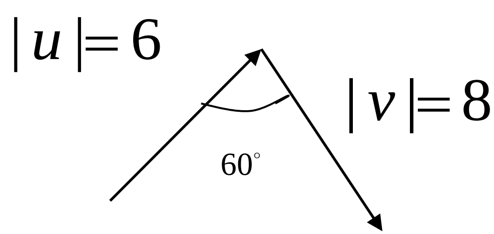
   (i) $\underline{a} = (-2,3,4), \underline{b} = (3,0,1)$

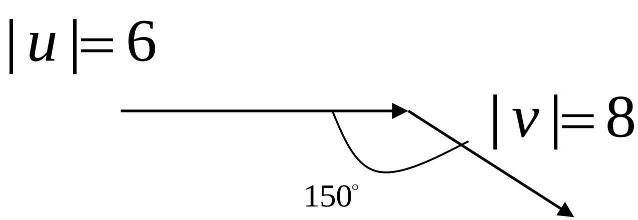
   (ii) $\underline{a} = (1,2,-3), \underline{b} = (5,-2,-1)$

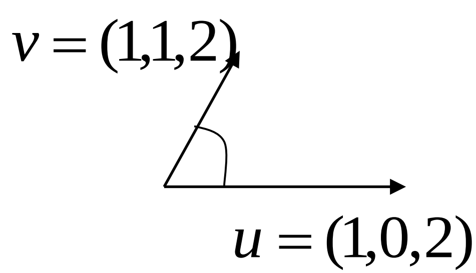
   (iii) $\underline{a} = (1,2,0), \underline{b} = (0,2,4)$

2. Find $|\mathrm{u} \times \mathrm{v}|$ and determine whether $\mathrm{v} \times \mathrm{u}$ is directed into the page of out of page









3. Consider the function

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } (x, y) \neq (0,0) \\ 0, & \text{if } (x, y) = (0,0) \end{cases}$$

Is f continuous at (0,0)? why or why not?

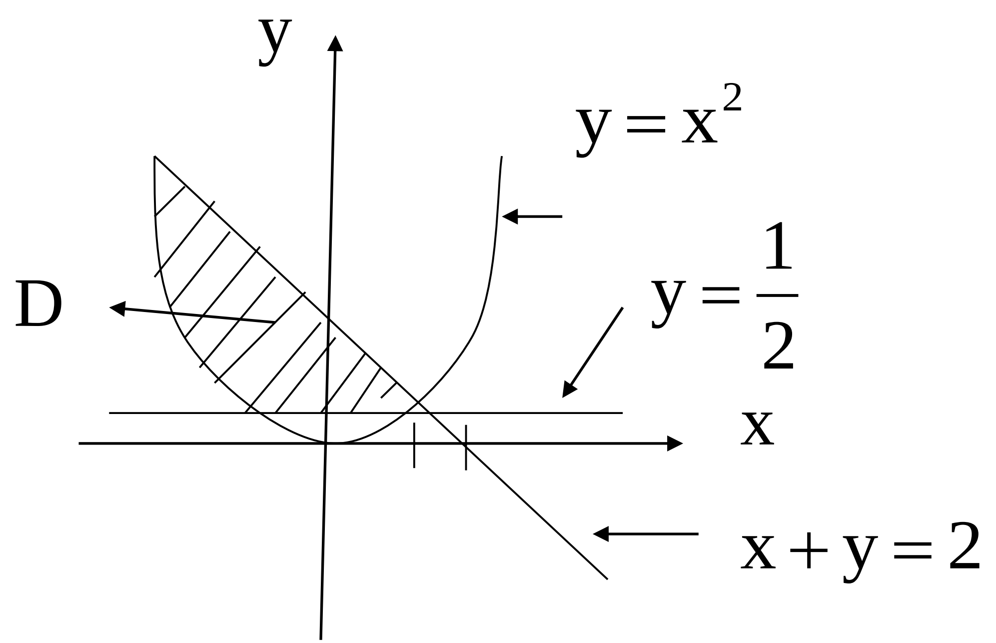
4. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0,0) \\ 1, & \text{if } (x, y) = (0,0) \end{cases}$$

(i) show that $f(x, y)$ is not continuous at (0,0)

(ii) Find $\lim\limits_{(x,y)\to(o,o)} f(x, y)$

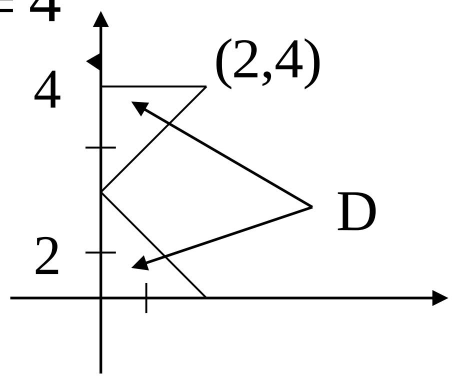
5. Evaluate $\iint_D xy\,dA$, where D is the region in the following



6. Evaluate $\iint_D \sin(xy^2)\,dA$, where $D = \{(x, y) \mid 1 \le x^2 + y^2 \le 4\}$

7. Evaluate $\iint_D x^2 dA$, where D is the region between the ellipse $x^2 + 2y^2 = 4$ and the circle $x^2 + y^2 = 4$

8. Evaluate $\iint_D xy^2 dA$



9. Evaluate the iterated integrals

(i) $\int_0^{\pi/2}\int_0^{\cos\theta} r^2 \sin\theta \, dr \, d\theta$ (ii) $\int_0^{\pi}\int_0^{1-\cos\theta} r\sin\theta \, dr \, d\theta$

(iii) $\int_0^{\pi}\int_0^{\sin\theta} r^2 \, dr \, d\theta$ (iv) $\int_0^{\pi}\int_0^{\sin y} x^2 \, dx \, dy$

10. Evaluate $\iint_D e^{x^2+y^2}\,dA$, where D is the region enclosed by

$x^2 + y^2 = 4$

11. Evaluate

(i) $\int_0^1 \int_0^{\sqrt{1-y^2}} \text{Sin}(x^2 + y^2)\,dxdy$

(ii) $\int_1^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)^{-\frac{1}{2}}\,dydx$

(iii) $\iint_D \text{Sin}\sqrt{x^2 + y^2}$, $\iint_D |\,\text{Sin}\sqrt{x^2 + y^2}\,|\,dA$, where

$D = \{(x, y) \mid x^2 + y^2 \le 4\pi^2\}$

12. Switch to rectangular coordinates and then evaluate

$\int_{3\pi/4}^{4\pi/3} \int_0^{-5\text{Sec}\theta} r^3 \text{Sin}^2\theta\,dr\,d\theta$

13. Find the area of the surface $z = x^2 + y^2$ below the plane $z = 9$

14. Find the area of the surface $x = uv,\ y = u + v,\ z = u - v,\ u^2 + v^2 \leq 1$

15. Find the area of the surface obtained by rotating the given curve about the x-axis

(i) $y = x^3, 0 \leq x \leq 2$

(ii) $y = \sqrt{x}, 4 \leq x \leq 9$

16. Evaluate the iterated integral

(i) $\int_0^1 \int_0^z \int_0^y xyz\,dx\,dy\,dz$ (ii) $\int_0^{\pi} \int_0^2 \int_0^{\sqrt{4-z^2}} z\sin y\,dx\,dz\,dy$

(iii) $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^x yz\,dy\,dz\,dx$ (iv) $\int_0^1 \int_x^{2x} \int_0^{x+y} 2xy\,dz\,dy\,dx$

Definition

1. Let S be a smooth parametric surface and is given by the equation

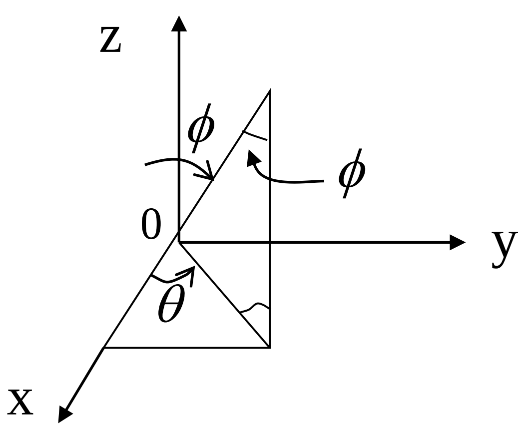
$$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k,\ \ (u, v) \in D$$

and S is covered just once as (u, v) ranges throughou t the parameter domain D, then the surface area of S is $A(S) = \iint_D |r_u \times r_v| \, dA$

where $r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k \qquad r_v = \frac{\partial x}{\partial v} i + \frac{\partial y}{\partial v} j + \frac{\partial z}{\partial v} k$

Example :

1. Spherical coordinate (ρ, θ, ϕ)



$p(x, y, z)$

$p(\rho, \theta, \phi)$

where $\rho = |\overline{op}| = \sqrt{x^2 + y^2 + z^2}$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta \qquad \rho \geq 0, 0 \leq \phi \leq \pi$

$z = \rho \cos\phi$

2. Find the rectangular coordinates   Spherical coordinate

$$(1, \frac{\pi}{4}, \frac{\pi}{3}) \qquad x = 1 \cdot \sin\frac{\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{6}}{4}$$

$$y = 1 \cdot \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{4}$$

$$z = 1 \cdot \cos\frac{\pi}{3} = \frac{1}{2}$$

3. Change from rectangular to spherical coordinates

(i) $(1,1,\sqrt{2})$ (ii) $(\sqrt{3},0,1)$ (iii) $(0,2\sqrt{3},-2)$

Sol:

$$\text{(i)}\ \rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = 2$$

$$z = \rho\cos\phi \Rightarrow \sqrt{2} = 2\cos\phi,\ \phi = \frac{\pi}{4}$$

$$x = \rho\sin\phi\cos\theta \Rightarrow 1 = 2\sin\frac{\pi}{4}\cos\theta,\ \theta = \frac{\pi}{4}$$

4. parametric surface S

(i) $S = \left\{ \begin{array}{l} (x, y, z) \mid x = 4\sin\phi\cos\theta,\ y = 4\sin\phi\sin\theta,\ z = 4\cos\phi \\ \qquad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi \end{array} \right\}$

S - The surface of a spherical of radius 4

(ii) Find the surface area of S

Sol : The parametric equation is

$$r(\theta, \phi) = 4\sin\phi\cos\theta \cdot i + 4\sin\phi\sin\theta \cdot j + 4\cos\phi \cdot k$$

and $(\theta, \phi) \in D = \{(\theta, \phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$

caculate $r_\theta = -4\sin\phi\sin\theta \cdot i + 4\sin\phi\cos\theta \cdot j + 0 \cdot k$

$$r_\phi = 4\cos\phi\cos\theta \cdot i + 4\cos\phi\sin\theta \cdot j + 4(-\sin\phi) \cdot k$$

$$r_\theta \times r_\phi = \begin{vmatrix} i & j & k \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \end{vmatrix}$$

$$= -16\sin^2\phi\cos\theta i - 16\sin^2\phi\sin\theta j + 16\cos\phi\sin\phi k$$

Thus

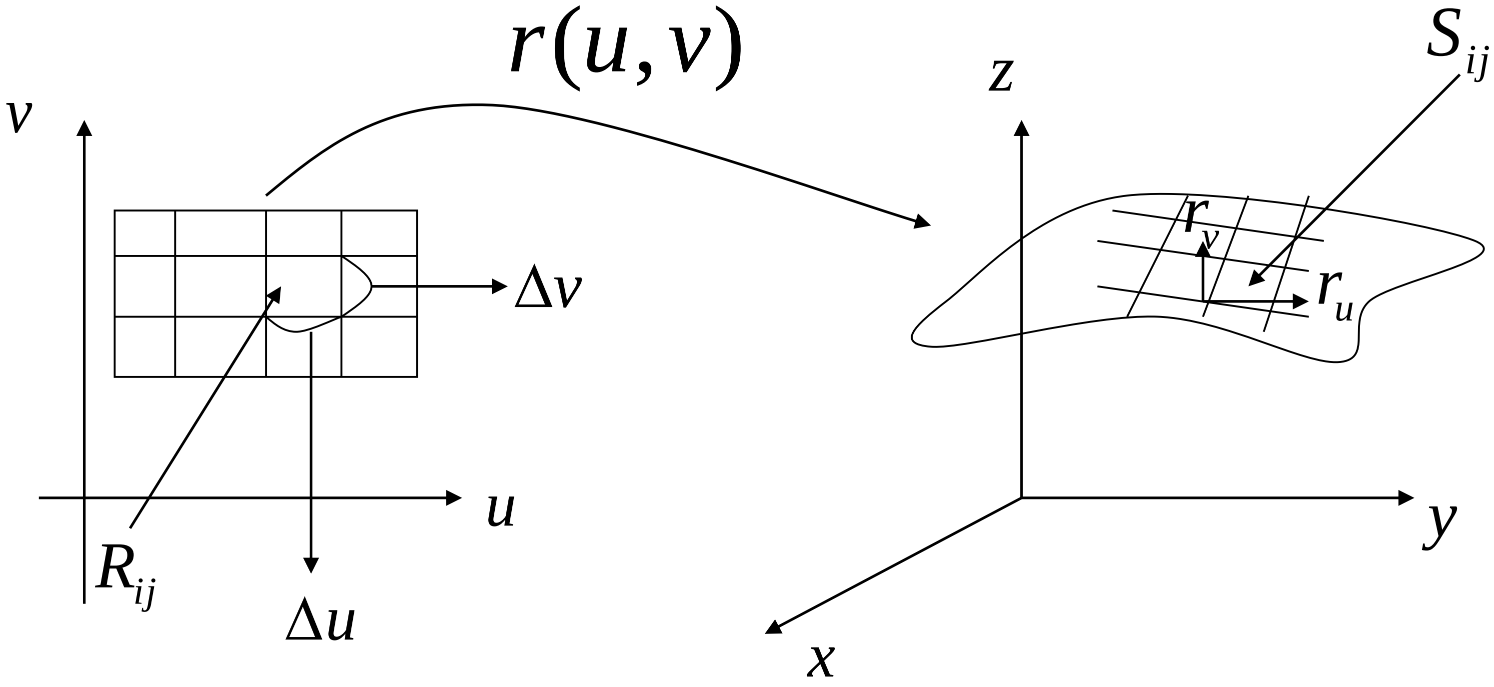
$$|\mathrm{r}_\theta \times \mathrm{r}_\phi| = \sqrt{(-16\sin^2\phi\cos\theta)^2 + (-16\sin^2\phi\sin\theta)^2 + (16\cos\phi\sin\phi)^2}$$

$$= 16\sin\phi$$

$$\mathrm{A(S)} = \iint_D |\mathrm{r}_\theta \times \mathrm{r}_\phi| \, \mathrm{dA} = \int_0^\pi \int_0^{2\pi} 16\sin\phi \, \mathrm{d}\theta \, \mathrm{d}\phi = 4\pi \cdot 16$$

Remark

1. $\mathrm{A(S)} = \iint_D |\mathrm{r_u} \times \mathrm{r_v}| \, \mathrm{dA}$



The area of S_{ij}

$$A(S_{ij}) \approx |\Delta u \cdot r_u \times \Delta v \cdot r_v| = |r_u \times r_v|\Delta u \Delta v$$

$$A(S) = \sum A(S_{ij}) \approx \sum (r_u \times r_v) \cdot \Delta u \Delta v$$

$$\to \iint_D |r_u \times r_v| dA$$

2. $S = \{(x, y, z) \mid z = f(x, y), (x, y) \in D\}$

The surface area of S is

$$A(S) = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dA$$

$(\because r(x, y) = xi + yj + f(x, y)k)$

Triple Integrals

Rectangular box :

$$B = \{(x, y, z) \mid a \le x \le b, c \le y \le d, e \le z \le f\}$$

$$= [a, b] \times [c, d] \times [e, f]$$

Example :

1. $B = [0,1] \times [1,3] \times [0,2]$

Definition :

Let $B = [a, b] \times [c, d] \times [e, f]$ be a rectangular box. $[a, b]$ is divided into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, $[c, d]$ is a divided into m subintervals $[[y_{j-1}, y_j]$ of equal width Δy, $[e, f]$ is a divided into n subintervals $[[z_{k-1}, z_k]$ of equal width Δz

1. $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$

2. The volume of $B_{ijk} \rightarrow \Delta v = \Delta x \cdot \Delta y \cdot \Delta z$

3. The triple Riemann sum $\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta v$

4. The triple intergral of f over the box B is

$$\iiint_B f(x, y, z)dv = \lim_{l,m,n \rightarrow 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta v$$

if this limit exists

Theorem(Fubini' s Theorem)

If f is continuous on $B = [a, b] \times [c, d] \times [e, f]$

then $\iiint_B f(x, y, z)dv = \int_e^f \int_c^d \int_a^b f(x, y, z)dxdydz$

Example :

1. Evaluate $\iiint_B xyz^2 dv$, where $B = [0,1] \times [1,2] \times [1,2]$

2. Evaluate $\iiint_B (x + yz)dv$, where $B = [-1,1] \times [1,3] \times [0,2]$

3. $E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 2, x \le z \le 1 - x - y\}$

How to define $\iiint_E x^2 yzdv = ?$

For a general bounded region E. Consider a rectangular box

$$B \supset E, \text{ and define } F(x, y, z) = \begin{cases} f(x, y, z) & \text{if}(x, y, z) \in E \\ 0 & \text{if}(x, y, z) \notin E \end{cases}$$

$$\text{Define } \iiint_E f(x, y, z)dv = \iiint_B F(x, y, z)dv$$

properties

1. If $E = \{(x, y, z) \mid (x, y) \in D, \phi_1(x, y) \le z \le \phi_2(x, y)\}$

$$\text{then } \iiint_E f(x, y, z)dv = \iint_D \left(\int_{\phi_{1(x,y)}}^{\phi_{2(x,y)}} f(x, y, z)dz \right) dA$$

2. If $E = \{(x, y, z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), \phi_1(x, y) \le z \le \phi_2(x, y)\}$

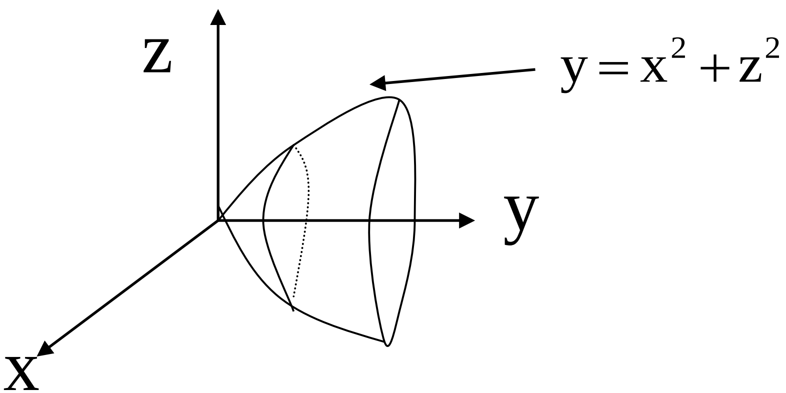
$$\text{then } \iiint_E f(x, y, z)dv = \int_a^b \int_{g_{1(x)}}^{g_{1(x)}} \int_{\phi_{1(x,y)}}^{\phi_{2(x,y)}} f(x, y, z)dzdydx$$

Examle :

1. $E = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$

$$\iiint_E z\,dv = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

2. $E = \{(x, y, z) \mid -2 \le x \le 2, x^2 \le y \le 4, -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}$



$$E = \{(x, y, z) \mid -2 \le x \le 2, -\sqrt{4 - x^2} \le z \le \sqrt{4 - x^2}, z^2 + x^2 \le y \le 4\}$$

$$\iiint_E \sqrt{z^2 + x^2}\,dv = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z^2+x^2}^{4} \sqrt{z^2 + x^2}\,dy\,dz\,dx = \frac{128\pi}{15}$$